

# Stabilization Methods for Spectral Element Computations of Incompressible Flows

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A stabilization method for the spectral element computation of incompressible flow problems is investigated. It is based on a filtering procedure which consists in filtering the velocity field by a spectral vanishing Helmholtz-type operator at each time step. Relationship between this filtering procedure and SVV-stabilization method, introduced recently in [JCP, 2004, **196**(2), p680], is established. A number of numerical examples are presented to show the accuracy and stabilization capability of the method.

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**KEY WORDS:** Spectral element method; stabilization; Navier–Stokes equations.

## 1. INTRODUCTION

Despite the success of the spectral element methods (SEM) in the applications of, among many examples, incompressible flows, severe stability problems have also been encountered in the past, especially when facing problems having weak physical diffusion. These problems result from the fact that spectral approximations are much less numerically diffusive than low-order ones, even minor errors and under resolution can make the calculation unstable. For a long time, numerous filtering techniques have been proposed to overcome the stability problem. In the frame of spectral element approximations it is however, essential to preserve the inter-element continuity, as discussed in [1]. One of the most recent advances in this direction has been proposed in [2], where an interpolation-based stabilization procedure is applied at the end of each time step of the Navier–Stokes discretization. This procedure was then interpreted in [10] as a filter if  $\{L_i - L_{i-2}, 2 \leq i \leq N\}$  basis is used to expand the velocity.

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Another stabilization technique has been recently proposed by Xu and Pasquetti in [14], where the spectral vanishing viscosity (SVV) method was integrated into the weak formulation of the Navier–Stokes equations. This technique has been proven to be an efficient stabilization method possessing the properties of the inter-element continuity and the spectral accuracy. The SVV method was initially developed for the resolution of hyperbolic equations using the spectral methods [5, 7, 12]. Recently, it has been suggested to use the SVV method, using modal basis [6] or in the frame of collocation methods in a Cartesian geometry [9], for the large-eddy simulation of turbulent flows. However, for the first time we have generalized in [14] the SVV idea to the multi-dimensional case with general deformed elements using nodal basis.

The present work follows the subject of [14], and try to give an interpretation of the SVV-SEM as a filtering technique, the latter has the advantage of being simpler to use, without the need to modify the existing SEM code. To this end:

First, we recall the SVV stabilized SEM introduced in [14], then try to interpret it as a filtering procedure. We show how to implement the filtering technique in the frame of a Navier–Stokes spectral element solver.

Second, we show that the filtering procedure is a reasonable approximation to the SVV-SEM, and that in certain case, the two approaches are strictly equivalent.

Third, we consider an analytical solution to study the accuracy property. We will see that the filtering procedure not only stabilizes the calculation, but also improves the accuracy of the overall schema.

Finally, in order to numerically demonstrate the stabilization property of the method, we compute the shear layer roll-up problem at Reynolds numbers  $Re = 10^5$  and two dimensional (2D) Von-Karman flows behind a cylinder at  $Re = 1000$ .

## 2. STABILIZED SPECTRAL ELEMENT FORMULATION

The spectral element approximation of the incompressible Navier–Stokes equations yields the following semi-discrete problem [8], to be solved at each time-step after the time-discretization: Find  $u_N \in X_N$  and  $p_N \in M_N$  such that

$$\begin{cases} (D_t u_N, v_N) + \nu(\nabla u_N, \nabla v_N) - (\nabla \cdot v_N, p_N) = (s_N, v_N), & \forall v_N \in X_N, \\ (\nabla \cdot u_N, q_N) = 0, & \forall q_N \in M_N \end{cases} \quad (1)$$

where  $u_N$ ,  $p_N$  and  $s_N$  denote the approximations of the velocity, pressure and source term, respectively,  $D_t u_N$  the material derivative of  $u_N$ ,  $\nu$  the

dimensionless viscosity and  $(\cdot, \cdot)$  the standard  $L^2(\Omega)$  inner product. The domain  $\Omega$ , assumed to be 2D for the sake of simplicity, is partitioned into a conforming decomposition

$$\bar{\Omega} = \bigcup_{k=1}^K \bar{\Omega}^k, \quad \Omega^k \cap \Omega^l = \emptyset, \quad \forall k, l, k \neq l.$$

The discrete velocity space  $X_N$  and pressure space  $M_N$  consist of,

$$X_N = P_{N,K}(\Omega)^2 \cap H_0^1(\Omega)^2, \quad M_N = P_{N-2,K}(\Omega) \cap L_0^2(\Omega)$$

with standard notations for the Hilbert spaces  $H_0^1(\Omega)$  and  $L_0^2(\Omega)$ , and with:

$$P_{N,K}(\Omega) = \{v \in L^2(\Omega); v|_{\Omega^k} \circ f^k \in P_N(\Lambda^2), 1 \leq k \leq K\},$$

where  $f^k$  is the transformation function from the reference domain  $\Lambda^2$ , with  $\Lambda = (-1, 1)$ , to  $\Omega^k$  and  $P_N$  the space of the polynomials of maximum degree  $N$  in each variable. For the reason of simplicity again, homogeneous boundary conditions have been assumed through the use of the  $H_0^1(\Omega)$  space.

In [14], a stabilized spectral element approximation of problem (1) was introduced and analyzed. The proposed method consists in adding a stabilization term  $V_N$  in (1), yielding the following weak formulation

$$\begin{cases} (D_t u_N, v_N) + V_N(\nabla u_N, \nabla v_N) + v(\nabla u_N, \nabla v_N) - (\nabla \cdot v_N, p_N) \\ = (s_N, v_N), \quad \forall v_N \in X_N, \\ (\nabla \cdot u_N, q_N) = 0, \quad \forall q_N \in M_N \end{cases} \tag{2}$$

where  $V_N(\nabla u_N, \nabla v_N)$  is a SVV term written in weak form.

Let  $f$  be the mapping from  $(X, Y)$ , in the reference domain  $\Lambda^2$ , to  $(x, y)$  in the spectral element  $\Omega^k$  and  $g = f^{-1}$ ,  $G$  the transpose of the Jacobian matrix of  $g$  and  $J$  the Jacobian determinant of  $f$  (To simplify the notation we use here  $f, g, \dots$ , rather than  $f^k, g^k, \dots$ ). Then the definition of  $V_N$  used in [14] takes the symmetric bilinear form corresponding to the  $k$ -element:

$$V_N^k(\nabla u_N, \nabla v_N) = \epsilon_N(\tilde{G}Q^{1/2}(\tilde{\nabla} \tilde{u}_N), J\tilde{G}Q^{1/2}(\tilde{\nabla} \tilde{v}_N))_{L^2(\Lambda^2)}, \quad k = 1, \dots, K, \tag{3}$$

where  $\epsilon_N = O(1/N)$ ,  $\tilde{\nabla}$  denotes the gradient with respect to the reference domain and with  $\tilde{\varphi}_N = \varphi_N \circ f$ . The meaning of  $Q^{1/2}(\tilde{\nabla} \tilde{u}_N)$  makes use the 1D scalar definition of  $Q$ :

$$Q^{1/2}(\tilde{\nabla} \tilde{u}_N) = \begin{pmatrix} Q^{1/2}(\partial_X \tilde{u}_N(\cdot, Y)) \\ Q^{1/2}(\partial_Y \tilde{u}_N(X, \cdot)) \end{pmatrix},$$

where  $Q^{1/2}$  is the spectral viscosity operator such that, with  $L_i$  for the Legendre polynomial of degree  $i$ :

$$Q^{1/2}\phi \equiv \sum_{i=0}^N \sqrt{\hat{Q}_i} \hat{\phi}_i L_i, \quad \forall \phi: \phi = \sum_{i=0}^{\infty} \hat{\phi}_i L_i$$

with  $\hat{Q}_i = 0$  if  $i \leq m_N$  and

$$\hat{Q}_i = \exp\left(-\left(\frac{N-i}{m_N-i}\right)^2\right), \quad \text{if } i > m_N,$$

$m_N$  takes a suitable value between 0 and  $N$ .

Finally, global  $V_N$  is defined as the summation:

$$V_N(\nabla u_N, \nabla v_N) = \sum_{k=1}^K V_N^k(\nabla u_N, \nabla v_N). \tag{4}$$

The efficient implementation of the stabilized spectral element approximation (2) is quite technique, we refer to [14] for a reasoning for the definition (3) and its detailed implementation.

### 3. FILTER-BASED STABILIZING PROCEDURE

In this section, we will first introduce a filtering procedure, then try to give an interpretation of the SVV stabilization method presented in the previous section. We first go back to the standard  $P_N \times P_{N-2}$  spectral element semi-discrete problem (1) without stabilization. We introduce the following second-order rotational projection schema [3, 4] plus a filtering step to discretize in time the problem (1):

- Diffusion step (computation of the intermediate velocity):

Find  $u_N^* \in X_N$ , such that for all  $v_N \in X_N$ ,

$$\left( \frac{3u_N^* - 4\check{u}_N^n + \check{u}_N^{n-1}}{2\Delta t}, v_N \right) + \nu(\nabla u_N^*, \nabla v_N) = (\nabla \cdot v_N, p_N^n) + (s_N^{n+1}, v_N), \tag{5}$$

where  $\Delta t$  is the time step.  $\check{u}_N^n, \check{u}_N^{n-1}$  are the transport at different instant of the previous solution on the characteristics. A detailed description on their computation was given in [13]. We point out that the time derivative is approximated by the second-order backward differentiation.



active on the last highest modes and the “artificial viscosity”  $\frac{2\Delta t}{3}v_{\in N}$  is generally small, the filtering problem (6) results in a well-conditioned symmetric positive definite system, hence can be inverted by standard elliptic solvers like conjugate gradient algorithm.

### 3.1. Relationship with the SVV Formulation

It can be shown that the filtering procedure (5)–(7) is a reasonable approximation to the SVV formulation (2). To see that, we need to establish the equations satisfied by the filtered velocity.

Combining (5) and (6), we have

$$\begin{aligned} & \left( \frac{3\bar{u}_N^* - 4\check{u}_N^n + \check{u}_N^{n-1}}{2\Delta t}, v_N \right) + V_N(\nabla\bar{u}_N^*, \nabla v_N) + \nu(\nabla\bar{u}_N^*, \nabla v_N) \\ & - \frac{2\Delta t}{3}v_{\in N}(\nabla\nabla \cdot \mathbf{Q}\nabla\bar{u}_N^*, \nabla v_N) = (\nabla \cdot v_N, p_N^n) + (s_N^{n+1}, v_N). \end{aligned} \quad (8)$$

If we ignore the high order term  $\frac{2\Delta t}{3}v_{\in N}(\nabla\nabla \cdot \mathbf{Q}\nabla\bar{u}_N^*, \nabla v_N)$ , then combining (8) and (7) gives

$$\begin{aligned} & \left( \frac{3\bar{u}_N^* - 4\check{u}_N^n + \check{u}_N^{n-1}}{2\Delta t}, v_N \right) + V_N(\nabla\bar{u}_N^*, \nabla v_N) + \nu(\nabla\bar{u}_N^*, \nabla v_N) \\ & = (\nabla \cdot v_N, p_N^n) + (s_N^{n+1}, v_N), \\ & \begin{cases} \left( \frac{3u_N^{n+1} - 3\bar{u}_N^*}{2\Delta t}, v_N \right) + (\nabla(p_N^{n+1} - p_N^n + \nu\nabla \cdot \bar{u}_N^*), v_N) = 0, \\ (u_N^{n+1}, \nabla q_N) = 0, \end{cases} \end{aligned}$$

which is nothing else than the second-order rotational projection schema of the stabilized spectral element formulation (2).

It can be easily shown that in the Fourier case  $-(\nabla\nabla \cdot \mathbf{Q}\nabla\bar{u}_N^*, \nabla v_N)$  is purely diffusive, hence helps to stabilizing the schema. In the general case (not periodic), we are unable to prove the diffusibility, however, we note that this additional term has a small magnitude tending to zero when  $\Delta t v_{\in N}$  tends to zero. Hence we can expect that the difference on the stabilization capability between the proposed filtering procedure and SVV-SEM is negligible. This point will be confirmed through the computation of the shear layer roll-up problem and 2D high Reynolds number wake of a cylinder.

We should note that in the case where the physical diffusion is absent (i.e.  $\nu = 0$ ), the filtered-SEM and SVV-SEM are strictly equivalent.

## 4. NUMERICAL RESULTS

The purpose of this section is to investigate the convergence and stability property of the filtered-SEM. To this end we first consider the Navier–Stokes equations in  $\Lambda^2$ , with a stiff analytical solution, to demonstrate that the filtered-SEM not only keeps exponentially accurate, but also improves the accuracy as compared to the unfiltered case. Then the shear layer roll-up problem and Von-Karman flow past a cylinder is computed to show the stabilization capability of the method.

### 4.1. An Analytical Solution

In this first test,  $\nu$  is fixed to be  $10^{-2}$ . In order to check numerically the effect of the filtering procedure (6) on the spectral element solution, we choose the analytical exact solution:

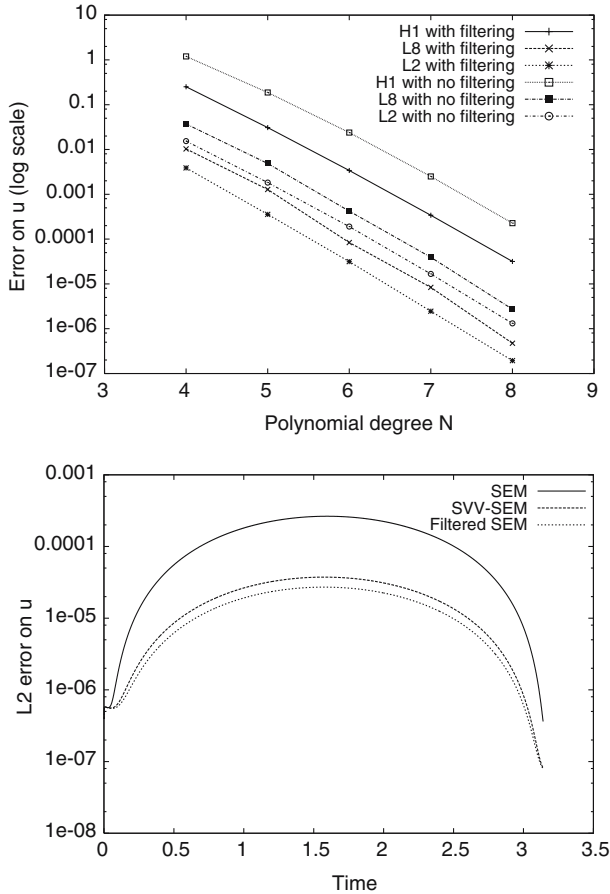
$$\begin{cases} u_1(x, y, t) = \sin(2\pi x) \cos(2\pi y) \sin(t) \\ u_2(x, y, t) = -\cos(2\pi x) \sin(2\pi y) \sin(t) \end{cases}$$

and the computational domain  $\Lambda^2$  is partitioned into  $10 \times 10$  square elements.

Figure 1(a) shows some velocity errors at  $t=1$  in  $H^1, L^\infty$  and  $L^2$  norms obtained with the filtered-SEM and the non-filtered case respectively. Here, as motivated by the numerical experiences performed in [14], the filtering is applied with parameters  $m_N = N - 2$  and  $\epsilon_N = 1/N$ . Surprisingly, contrast to our previous results [14] obtained by solving the *steady* Helmholtz equations, the stabilized-SEM not only preserves the exponential convergence rate (the errors show an exponential decay when the polynomial degree is increased), but also is *more* accurate than the standard (non-filtered) SEM.

In order to know whether the SVV-SEM introduced in [14] possesses similar property, this time-dependent solution is also calculated by using SVV-SEM. The  $L^2$ -errors on  $u$  as a function of time  $t$  is plotted in Fig. 1(b), showing that more accurate solution is obtained by both SVV-SEM and filtered-SEM, and that the difference between the SVV-SEM and filtered-SEM is insignificant. This confirms our analysis given in Sec. 3.

The non-linear convective term may be responsible for different behaviors between the steady Helmholtz equations and unsteady Navier–Stokes equations: in the latter case the spurious “high-frequency modes” resulting from aliasing effects may be damped when the SVV or filtering is activated, resulting in better results.



**Fig. 1.** (Top) Errors on  $u$  in the  $H^1$ ,  $L^\infty$  and  $L^2$  norms as a function of the polynomial degree  $N$ . (Bottom)  $L^2$  Errors on  $u$  as a function of the time obtained without filtering and with filtering.

### 4.2. Applications

In this subsection the efficiency of the proposed filtered SEM is demonstrated through the computation of the shear layer roll-up problem and 2D high Reynolds number Von-Karman flow behind a cylinder.

#### 4.2.1. Shear Layer Roll-Up Problem

As a well-known test problem [2], shear layer roll-up flow is computed in the domain  $[0, 1]^2$ , with doubly-periodic boundary conditions and following initial conditions

$$u_1(x, y, t) = \begin{cases} \tanh(30(y - 0.25)) & \text{for } y \leq 0.5 \\ \tanh(30(0.75 - y)) & \text{for } y > 0.5 \end{cases}, \quad u_2(x, y, t) = 0.05 \sin(2\pi x).$$

In all case, the domain is broken into  $16 \times 16$  equal square elements. The Reynolds number is defined by  $Re = \frac{Lu_{\max}}{\nu}$ , where  $L$  and  $u_{\max}$  are respectively the dimensionned side length and maximum training velocity. Figure 2(a) shows the vorticity for Reynolds number  $Re = 10^5$  at  $t = 1.055$  for  $N = 8$ ,  $t = 1.035$  for  $N = 16$ , just prior to blow up for the standard SEM. Increasing  $N$  and  $K$  and decreasing  $\Delta t$  up to reasonable values do not help to stabilize the simulation, as already indicated in [2]. Filtering the intermediate velocity dramatically improves the stability, as shown in Fig. 2(c), where vorticity contours at  $t = 1.5$  are plotted for filtered SEM with  $N = 8$  and  $N = 16$ , respectively. In order to avoid the insignificant zero contour, an even number of contour levels has been used. Also shown in Fig. 2(b) is the result by SVV-SEM: no significant difference is observed as compared with the filtered SEM.

#### 4.2.2. Von-Karman Flows Past a Cylinder

Several calculations have been carried out for the flow around an impulsively started circular cylinder, for various Reynolds numbers. Our goal here is to check the capabilities of the filtered-SEM.

In all our calculations, the number of the elements has been fixed to  $K = 310$ . The Reynolds number is defined as  $Re = U_{\infty}D/\nu$ , where  $U_{\infty}$  is the free stream velocity and  $D$  is the cylinder diameter. As already pointed out in [14], in this domain decomposition, using the standard SEM we were unable to compute flows at  $Re \geq 500$  at any reasonable resolution.

The stabilization effect of the filtering is checked by the long time simulation of the unsteady wake at  $Re = 1000$ . The used polynomial degree corresponding to each macro-element is  $N = 12$  and the time step equals 0.02, using 5 sub time-cycles in the transport step. As expected, with the filtering stabilization, we are now able to carry out long time simulation. The computed flow structures captured at  $t = 100$  (no shown here) compare well with the ones given in [14] obtained by the SVV-SEM. By observing the time evolution of the cross-flow velocity at a selected point in the wake, we obtain a Strouhal number of 0.25. This result is also in good agreement with that computed by the SVV-SEM or given in [11].

In summary, we can draw from the above numerical tests the following conclusions:

- The filtered-SEM not only remains spectrally convergence, but also is more accurate than the standard SEM when applied to the unsteady Navier–Stokes problems;

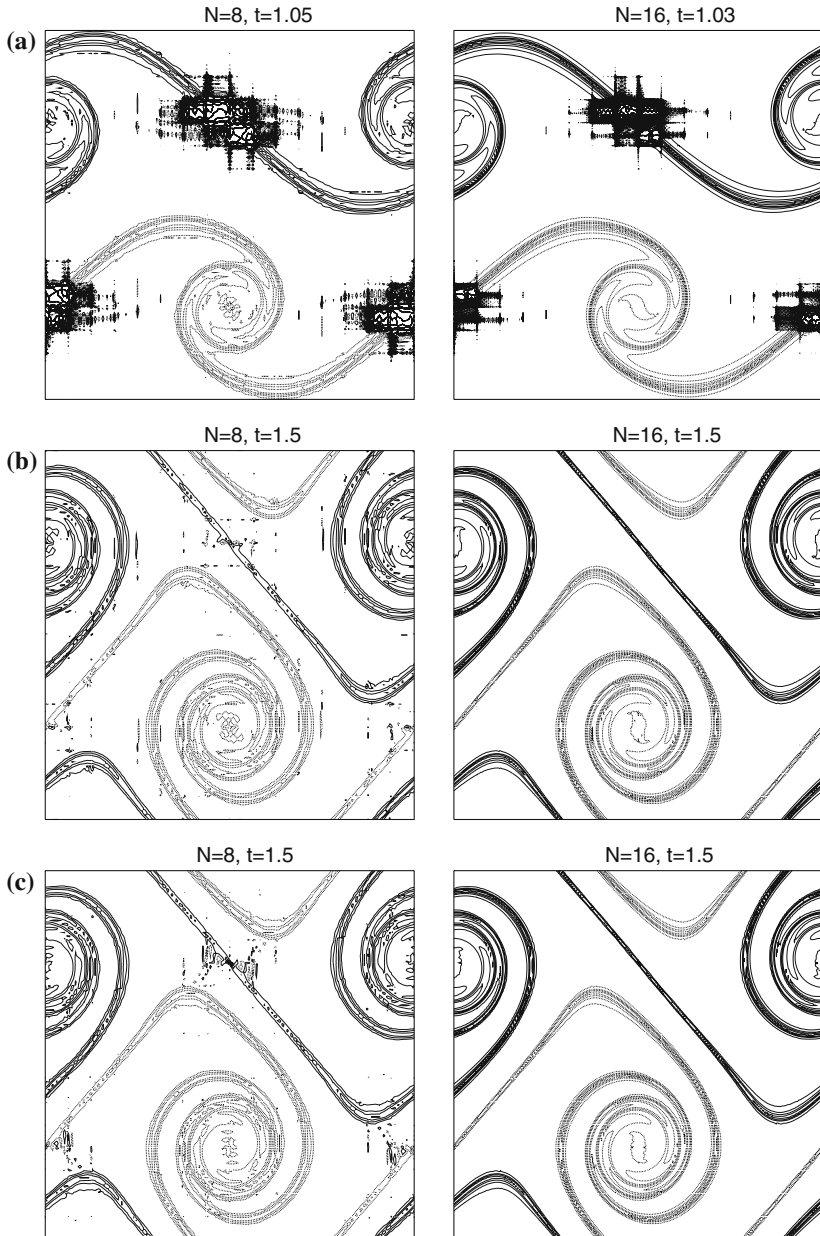


Fig. 2. Vorticity contours ( $-70 < \omega < 70$ , 14 equidistant levels) for different methods: (a) Standard SEM; (b) SVV-SEM; (c) Filtered-SEM.

- The filtering procedure is also an efficient stabilization method as compared to our previously proposed SVV-SEM. It allows to compute flows of high Reynolds number where simulations by the standard SEM fail to converge.

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## REFERENCES

1. Boyd, J. P. (1998). Two comments on filtering (artificial viscosity) for Chebyshev and Legendre spectral and spectral element methods: Preserving the boundary conditions and interpretation of the filter as a diffusion. *J. Comput. Phys.* **143**, 283–288.
2. Fischer, P., and Mullen, J. (2001). Filter-based stabilization of spectral element methods. *C. R. Acad. Sci. Paris* **332**(1), 265–270.
3. Guermond, J. L., and Shen, J. (2004). On the error estimates for the rotational pressure-correction projection methods. *Math. Comp.* **73**, 1719–1737.
4. Huang, F. H., and Xu, C. J. (2005). On the error estimates for the rotational pressure-correction projection spectral methods for the unsteady Stokes equations. *J. Comput. Math.* **23**(3), 285–304.
5. Kaber, O. S. M. (1996). A Legendre pseudospectral viscosity method. *J. Comput. Phys.* **128**, 165–180.
6. Karamanos, G. S., and Karniadakis, G. E. (2000). A spectral vanishing viscosity method for large-eddy simulation. *J. Comput. Phys.* **163**, 22–50.
7. Maday, Y., Kaber, O. S. M., and Tadmor, E. (1993). Legendre pseudo-spectral viscosity method for nonlinear conservation laws. *SIAM J. Numer. Anal.* **30**(2), 321–342.
8. Maday, Y., and Patera, A. T. (1988). Spectral element methods for the Navier–Stokes equations. In Noor, A. K. (ed.), *State-of-the-Art Surveys in Computational Mechanics* ASME, New York, pp. 71–143.
9. Pasquetti, R., and Xu, C. J. (2002). High-order algorithms for large eddy simulation of incompressible flows. *J. Sci. Comp.* **17**(1–4), 273–284.
10. Pasquetti, R., and Xu, C. J. (2002). Comments on “Filter-based stabilization of spectral element methods”. *J. Comput. Phys.* **182**, 646–650.
11. Qian, L., and Vezza, M. (2001). A Vorticity-based method for incompressible unsteady viscous flows. *J. Comput. Phys.* **172**, 515–542.
12. Tadmor, E. (1989). Convergence of spectral methods for nonlinear conservation laws. *SIAM J. Numer. Anal.* **26**(1), 30–44.
13. Xu, C. J., and Pasquetti, R. (2001). On the efficiency of semi-implicit and semi-Lagrangian spectral methods for the calculation of incompressible flows. *Inter. J. Numer. Meth. Fluids* **35**, 319–340.
14. Xu, C. J., and Pasquetti, R. (2004). Stabilized spectral element computations of high Reynolds number incompressible flows. *J. Comput. Phys.* **196**(2), 680–704.